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SOLUTIONS OF THE EQUATION OF HELMHOLTZ IN A QUARTER PLANE WITH AN OBLIQUE BOUNDARY CONDITION

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1. Introduction.

In the hydrodynamical studies of the motion of the North Sea which were initiated at the Mathematical Centre in 1953 a rectangular model was chosen to represent the North Sea basin. Neglecting amongst others the influence of the Channel and the Kattegat this rectangle is bounded on three sides by coasts and on the fourth side by an infinitely deep ocean. It can be expected that corners where different types of boundary conditions meet may cause analytical difficulties in the form of singularities (see LAUWERIER [1] and [2]). In the present report a simplified model is studied in order to obtain further insight in the peculiarities of such corners. For this purpose the geometry of the problem has been reduced to a quarter plane with a boundary condition of the coast type and one of the ocean type.

The model describing the hydrodynamical motion of a quarterplane sea with the above-mentioned boundaries is as follows

(1.1) $$\Delta u - p^2 u = 0 , \qquad x > 0, y > 0,$$

(1.2) $$\begin{cases} u = 0 , & x = 0, y > 0 \text{ ("coast" condition)}, \\ \dfrac{\partial u}{\partial y} + k \dfrac{\partial u}{\partial x} = -\psi(x) , & x > 0, y = 0 \text{ ("ocean" condition)}, \end{cases}$$

where $u = u(x,y)$ is the stream function,

$\psi = \psi(x)$ is a given real function which satisfies a uniform Hölder contion for $0 \leq x \leq \infty$,
$k = \mathrm{tg}\mu\pi$ $(|\mathrm{Re}\ \mu| \leq \frac{1}{2})$ is a known constant

and p is a known real constant.
The constant k depends upon the value of the force of Coriolis and the bottom friction.

Solutions of (1.1) can be represented by the integral

$$u = \int_0^\infty \alpha^{-1} f(\alpha) \sin \alpha x \, e^{-\alpha\gamma y} \, d\alpha ,$$

where $f(\alpha)$ is an unknown real function and $\gamma = \gamma(\alpha) = [1 + (p/\alpha)^2]^{1/2}$, Re $\gamma \geq 0$.

The factor α^{-1} is added to simplify the ensuing relations. The solution (1.3) satisfies the boundary condition at $x = 0$ and is bounded at infinity. The function $f(\alpha)$ can be determined with the help of the condition at $y = 0$.

Substituting (1.3) in this condition we obtain an integral equation of an unusual type viz.

$$(1.4) \qquad \int_0^\infty f(\alpha)\,[\gamma(\alpha)\sin\alpha x - k\cos\alpha x]\,d\alpha = \psi(x), \qquad x > 0.$$

In the following we will distinguish the cases $p = 0$ and $p \neq 0$. In the first case equation (1.1) is the potential equation. With $\gamma(\alpha) \equiv 1$ formula (1.4) simplifies to

$$(1.5) \qquad \int_0^\infty f(\alpha)\sin(\alpha x - \mu\pi)\,d\alpha = \psi(x)\cos\mu\pi, \qquad x > 0.$$

If $p \neq 0$ we substitute $p = 1$ in all equations concerned. This simply means that we take p^{-1} as unit of length, so it causes no loss of generality. We further take $F(\alpha) = f(\alpha)\,\gamma(\alpha)$ and obtain

$$(1.6) \qquad \int_0^\infty F(\alpha)\,[\sin\alpha x - \gamma^{-1}(\alpha)\,k\cos\alpha x]\,d\alpha = \psi(x), \quad x > 0,$$

where $\gamma(\alpha) = [1 + \alpha^{-2}]^{1/2}$.

It is not difficult to find an explicit solution of (1.5) as is shown in sections 2 and 3 with the aid of respectively the Hilbert problem technique and the Mellin transform method. The two solutions thus obtained are equivalent. Equation (1.6) is inverted in section 4 with the aid of a combination of the above-mentioned techniques. The last section is devoted to an example.

2. The potential problem (Hilbert problem technique).

In this section we consider the inversion of equation (1.5)

$$\int_0^\infty f(\alpha)\sin(\alpha x - \mu\pi)\,d\alpha = \psi(x)\cos\mu\pi\ , \quad x > 0,$$

where μ is a known constant ($|\mathrm{Re}\,\mu| \leq 1/2$) and $\psi(x)$ is a given real Hölder continuous function.

We introduce the analytic function

$$(2.1) \qquad \Psi(z) = \int_0^\infty f(\alpha)\, e^{i\alpha z}\, d\alpha\ ,$$

where $z = x + iy$, $x \geq 0$, $y \geq 0$.
This function satisfies the conditions

$$(2.2) \qquad \begin{aligned} &\mathrm{Im}\{\Psi(z)\} = 0\ , && x = 0,\ y > 0, \\ &\mathrm{Im}\{e^{-\mu\pi i}\Psi(z)\} = \psi(x)\cos\mu\pi\ , && x > 0,\ y = 0, \end{aligned}$$

and $\Psi(z)$ is uniformly bounded in $x \geq 0$, $y \geq 0$.
By the conformal mapping $w = z^2$ ($w = u + iv$) the quarter plane $x \geq 0$, $y \geq 0$ is mapped upon the upper half plane $v \geq 0$. Taking $\Phi(w) = \Psi(w^{1/2})$ and $\phi(u) = \psi(u^{1/2})$ we obtain the following relations

$$(2.3) \qquad \begin{aligned} &\mathrm{Im}\{\Phi(u)\} = 0\ , && u < 0, \\ &\mathrm{Im}\{e^{-\mu\pi i}\Phi(u)\} = \phi(u)\cos\mu\pi\ , && u > 0, \end{aligned}$$

while $\Phi(w)$ is uniformly bounded in the upper half plane $v \geq 0$.

The relations (2.3) constitute a Hilbert problem for a half plane and are equivalent to

$$(2.4) \qquad \Phi^+(u) = e^{2\mu\pi i}\,\Phi^-(u) + 2ie^{\mu\pi i}\cos\mu\pi\,\phi(u), \quad u > 0,$$

where $\Phi^+(u) = \lim_{v\to+0}\Phi(w)$ and $\Phi^-(u) = \lim_{v\to-0}\Phi(w)$. The corresponding homogeneous problem ($\phi(u) \equiv 0$) has solutions which are linear combinations of factors $w^{\lambda-\mu}$, where λ is an integer. These functions tend to infinity either as $|w| \to 0$ or as $|w| \to \infty$. Therefore the homogeneous problem has no solution which is uniformly bounded for $v \geq 0$. The solution of the inhomogeneous problem must be unique and can be represented by the following integrals.

(2.5)

$$\Phi(w) = \frac{1}{\pi} e^{\mu\pi i} \cos \mu\pi \; w^{-\mu} \int_0^\infty \frac{t^\mu \phi(t)}{t-w} \, dt, \text{ if } -1/2 \leq \text{Re } \mu \leq 0,$$

or

$$\Phi(w) = \frac{1}{\pi} e^{\mu\pi i} \cos \mu\pi \; w^{1-\mu} \int_0^\infty \frac{t^{\mu-1} \phi(t)}{t-w} \, dt, \text{ if } 0 < \text{Re } \mu \leq 1/2.$$

Writing the Hölder condition for $\phi(\mu)$ at infinity as $|\phi(u)-\phi(\infty)| < C|\mu|^{-\alpha}$, $\alpha > 0$, it can be shown that the functions (2.5) have the prescribed behaviour as $|w| \to 0$ and $|w| \to \infty$, provided that $\phi(\infty) = 0$ if $\alpha < 1 + \text{Re } \mu$ for negative Re μ or if $\alpha < \text{Re } \mu$ for positive Re μ.

In the following we will only state the formulae for negative Re μ explicitly. Those for positive Re μ can then be deduced simply. In the original quarter plane

(2.6) $$\Psi(z) = \frac{2}{\pi} e^{\mu\pi i} \cos \mu\pi \; z^{-2\mu} \int_0^\infty \frac{s^{2\mu+1} \psi(s)}{s^2 - z^2} \, ds.$$

Now the solution of (1.5) can for example be written down with the aid of

(2.7) $$\text{Re}\{\Psi^+(x)\} = \int_0^\infty f(\alpha) \cos \alpha x \, d\alpha \; ,$$

where again $\Psi^+(x) = \lim_{y \to +0} \Psi(z)$.
It then follows that

(2.8) $$f(\alpha) = \frac{2}{\pi} \int_0^\infty \text{Re}\{\Psi^+(x)\} \cos \alpha x \, dx \; ,$$

where one can use for instance

(2.9) $$\text{Re}\{\Psi^+(x)\} = -\sin \mu\pi \cos \mu\pi \; \psi(x) + \frac{2}{\pi} \cos^2 \mu\pi \; x^{-2\mu} \int_0^\infty \frac{s^{2\mu+1} \psi(s)}{s^2 - x^2} \, ds.$$

As a check on the results we consider the limiting case $\mu = -1/2$. Provided that the limit $\psi^*(x) = \lim_{\mu \to -1/2} \psi(x) \cos \mu\pi$ exists, this corresponds to the boundary condition $(\partial u / \partial x) = -\psi^*(x)$ at $x > 0$, $y = 0$. Using representation (1.3) this yields immediately

$$f(\alpha) = -\frac{2}{\pi} \int_0^\infty \psi^*(x) \cos \alpha x \, dx .$$

When we let μ tend to $-1/2$ in (2.9) we get

$$\mathrm{Re}\{\Psi^+(x)\} = -\psi^*(x),$$

which can be substituted in (2.8) to give the same result.

We can modify (1.3) a little with the aid of (2.8) in order to be able to substitute (2.9) directly. If all integrals concerned converge absolutely so that we may change the order of integration we can write

$$\text{(2.10)} \qquad u = \frac{2}{\pi} \int_0^\infty \int_0^\infty \mathrm{Re}\{\Psi^+(t)\} \, \alpha^{-1} \sin \alpha x \cos \alpha t \, e^{-\alpha y} \, dt \, d\alpha$$

$$= \frac{2}{\pi} \int_0^\infty \mathrm{Re}\{\Psi^+(t)\} \, dt \int_0^\infty \alpha^{-1} \sin \alpha x \cos \alpha t \, e^{-\alpha y} \, d\alpha$$

$$= \frac{i}{2\pi} \int_0^\infty \mathrm{Re}\{\Psi^+(t)\} \left[\ln(t^2-z^2) - \ln(t^2-\bar{z}^2)\right] dt.$$

In section 5 a potential problem will be solved with the aid of the above formula.

3. The potential problem (Mellin transform method).

Anew we consider equation (1.5). We multiply both sides of the equation by $\sin(\beta x - \mu\pi)$ and integrate with respect to x from zero to infinity to obtain

$$(3.1)\quad f(\beta) - \frac{1}{\pi}\sin 2\mu\pi \int_0^\infty \frac{f(\alpha)}{\alpha+\beta}\,d\alpha = \frac{2}{\pi}\cos\mu\pi \int_0^\infty \psi(x)\sin(\beta x - \mu\pi)\,dx.$$

We then apply a Mellin transformation with respect to β in the following way

$$(3.2)\qquad F(s) = \int_0^\infty \beta^{s-1} f(\beta)\,d\beta ,$$

which gives rise to

$$(3.3)\quad \left[1 - \frac{\sin 2\mu\pi}{\sin s\pi}\right]F(s) = \frac{2}{\pi}\cos\mu\pi \int_0^\infty \psi(x)\,x^{-s}\,\Gamma(s)\sin(s-2\mu)\frac{\pi}{2}\,dx,$$

$$0 < \mathrm{Re}\, s < 1.$$

From this equation F(s) can be solved as

$$(3.4)\qquad F(s) = \frac{1}{\pi}\cos\mu\pi \int_0^\infty \psi(x)\,x^{-s}\,\Gamma(s)\frac{\sin s\pi}{\cos(s+2\mu)\frac{\pi}{2}}\,dx .$$

Next we apply an inverse Mellin transformation to (3.4) and again obtain different solutions for negative and positive values of μ.

$$f(\beta) = \frac{2}{\pi}\cos\mu\pi \int_0^\infty \psi(x)\,U_{2\mu+1}(2\beta x,0)\,dx, \text{ if } -1/2 \le \mathrm{Re}\,\mu \le 0,$$

(3.5)

and

$$f(\beta) = -\frac{2}{\pi}\cos\mu\pi \int_0^\infty \psi(x)\,U_{2\mu-1}(2\beta x,0)\,dx, \text{ if } 0 < \mathrm{Re}\,\mu \le 1/2,$$

where $U_\nu(2\beta x,0)$ denotes Lommel's function of two variables.

and $$U_{2\mu+1}(2\beta x,0) = \sin(\beta x-\mu\pi) + \frac{1}{\pi}\sin 2\mu\pi \int_0^\infty e^{-\beta x u}\frac{u^{-2\mu}}{u^2+1}\,du,$$

(3.6)

$$\text{if } -1/2 \le \mathrm{Re}\,\mu \le 0,$$

$$U_{2\mu-1}(2\beta x,0) = -\sin(\beta x-\mu\pi) + \frac{1}{\pi}\sin 2\mu\pi \int_0^\infty e^{-\beta xu} \frac{u^{2-2\mu}}{u^2+1}\,du,$$

if $0 < \mathrm{Re}\,\mu \leq 1/2$.

A notation which might be useful is obtained by introducing the Laplace transform Ψ of ψ

$$(3.7) \qquad \Psi(\beta u) = \int_0^\infty e^{-\beta xu}\,\psi(x)\,dx\ .$$

Again we only state explicitly the results for negative Re μ. Formula (3.5) takes the form

$$(3.8)\ f(\beta) = \frac{2}{\pi}\cos\mu\pi \int_0^\infty \psi(x)\sin(\beta x-\mu\pi)dx + \frac{4}{\pi^2}\sin\mu\pi\,\cos^2\mu\pi \int_0^\infty \Psi(\beta u)\,\frac{u^{-2\mu}}{u^2+1}\,du.$$

It is easy to prove that

$$\frac{2}{\pi}\int_0^\infty U_{2\mu\pm 1}(2\beta x,0)\,dx \int_0^\infty f(\alpha)\sin(\alpha x-\mu\pi)\,d\alpha = f(\beta).$$

One can also show that the formulae (3.5) and (3.8) are equivalent to the formulae (2.8).

Taking into consideration the properties of the known function $\psi(x)$ in any special case it should not be too difficult to choose from (2.8), (3.5) and (3.8) the representation of the solution which will yield the quickest results.

4. The Helmholtz problem.

We now proceed to consider equation (1.6)

$$\int_0^\infty F(\alpha)\,\left[\sin \alpha x - k\gamma^{-1}(\alpha)\cos \alpha x\right]\,d\alpha = \psi(x), \quad x > 0,$$

where $k = \operatorname{tg} \mu\pi$, $|\operatorname{Re} \mu| \leq 1/2$, is a known constant,

$$\gamma(\alpha) = (1 + \alpha^{-2})^{1/2}, \qquad \operatorname{Re} \gamma \geq 0,$$

and $\psi(x)$ is a given real uniformly Hölder continuous function.

If we substitute $s(x) = \int_0^\infty F(\alpha) \sin \alpha x \, d\alpha$ into the integral equation we get

$$(4.1) \qquad s(x) - \frac{2}{\pi} k \int_0^\infty \int_0^\infty \gamma^{-1}(\alpha)\, s(t) \sin \alpha t \cos \alpha x \, dt\, d\alpha = \psi(x), \; x > 0,$$

or, after some manipulation,

$$(4.2) \qquad s(x) - \frac{k}{\pi} \,\unicode{x2A0D}_0^\infty \frac{s(t)h_1(x,t)}{t-x}\, dt - \frac{k}{\pi} \int_0^\infty \frac{s(t)h_2(x,t)}{t+x}\, dt = \psi(x), \; x > 0,$$

where

$$(4.3) \qquad \begin{aligned} h_1(x,t) &= |t-x| \; K_1(|t+x|) \\ h_2(x,t) &= |t+x| \; K_1(|t+x|) \end{aligned}$$

and $K_1(x)$ is the first order modified Bessel function of the third kind. In order to conform to the notation of MUSKHELISHVILI [5] we introduce the function

$$(4.4) \qquad H(x,t) = \frac{h_1(x,t) - h_1(x,x)}{t-x} + \frac{h_2(x,t)}{t+x} \; ;$$

$$= \frac{|t-x| \; K_1(|t-x|) - 1}{t-x} + \frac{|t+x| \; K_1(|t+x|)}{t+x} \; ,$$

which has no poles in the interval of integration. In this way we obtain the singular integral equation

$$(4.5)\quad s(x) - \frac{k}{\pi} \int_0^\infty \frac{s(t)}{t-x}\, dt = \psi(x) + \frac{k}{\pi} \int_0^\infty s(t)\, H(x,t)\, dt, \qquad x > 0.$$

Following MUSKHELISHVILI we first solve the so-called "dominant equation"

$$(4.6)\qquad s(x) - \frac{k}{\pi} \int_0^\infty \frac{s(t)}{t-x}\, dt = \psi(x), \qquad x > 0,$$

by ordinary Hilbert technique.

If we consider the sectionally holomorphic function

$$(4.7)\qquad \Phi(z) = \frac{1}{2\pi i} \int_0^\infty \frac{s(t)}{t-z}\, dt$$

and we require $\Phi(z)$ to be uniformly bounded as $|z| \to 0$ and as $|z| \to \infty$, then equation (4.6) is equivalent to

$$(4.8)\qquad (1-ik)\, \Phi^+(x) - (1+ik)\, \Phi^-(x) = \psi(x), \qquad x > 0,$$

or else, with $k = \operatorname{tg} \mu\pi$,

$$(4.9)\qquad \Phi^+(x) = e^{2\mu\pi i}\, \Phi^-(x) + e^{\mu\pi i} \cos \mu\pi\, \psi(x), \qquad x > 0,$$

where $\Phi^+(x) = \lim_{y\to+0} \Phi(z)$ and $\Phi^-(x) = \lim_{y\to-0} \Phi(z)$.
This is nearly the same equation as (2.4) and its unique solution can be written down immediately as

$$\Phi(z) = \frac{1}{2\pi i}\, e^{\mu\pi i} \cos \mu\pi\, z^{-\mu} \int_0^\infty \frac{t^\mu \psi(t)}{t-z}\, dt, \quad \text{if } -1/2 \leq \operatorname{Re} \mu \leq 0,$$

(4.10)

$$\text{or}\qquad \Phi(z) = \frac{1}{2\pi i}\, e^{\mu\pi i} \cos \mu\pi\, z^{1-\mu} \int_0^\infty \frac{t^{\mu-1} \psi(t)}{t-z}\, dt, \text{ if } 0 < \operatorname{Re} \mu \leq 1/2.$$

We only state the results for $-1/2 \leq \operatorname{Re} \mu \leq 0$ in the following. The solution s(x) is obtained when we substitute (4.10) in

$$s(x) = \Phi^{+}(x) - \Phi^{-}(x) \quad \text{as}$$

$$(4.11) \qquad s(x) = \cos^2 \mu\pi . \psi(x) + \frac{1}{2\pi} \sin 2\mu\pi \; x^{-\mu} \fint_0^{\infty} \frac{t^{\mu}\psi(t)}{t-x} \, dt.$$

Now we replace $\psi(x)$ by $\psi(x) + \frac{k}{\pi} \int_0^{\infty} s(t)\, H(x,t)\, dt$. The resulting integral equation is a simple Fredholm equation of the second kind which is shown by MUSKHELISHVILI to be equivalent to equation (4.5). We get

$$(4.12) \qquad \sigma(x) - \frac{1}{\pi} \sin \mu\pi \int_0^{\infty} \frac{\sigma(t)}{t+x} \, dt = \theta(x), \qquad x > 0,$$

where $\sigma(x) = x^{\mu} s(x)$

and $$\theta(x) = \cos^2 \mu\pi \; x^{\mu} \; \psi(x) + \frac{1}{2\pi} \sin 2\mu\pi \fint_0^{\infty} \frac{t^{\mu}\psi(t)}{t-x} \, dt.$$

Equation (4.12) closely resembles (3.1), so we apply a Mellin transform with respect to x. Writing

$$S(\lambda) = \int_0^{\infty} x^{\lambda-1} \sigma(x)\, dx$$

(4.13)

and $$\Theta(\lambda) = \int_0^{\infty} x^{\lambda-1} \theta(x)\, dx,$$

we obtain

$$\left(1 - \frac{\sin \mu\pi}{\sin \lambda\pi}\right) S(\lambda) = \Theta(\lambda), \qquad 0 < \mathrm{Re}\, \lambda < 1.$$

We apply the inverse transformation to

$$S(\lambda) = \frac{\sin \lambda\pi}{\sin \lambda\pi - \sin \mu\pi} \, \Theta(\lambda)$$

and we get the result

$$(4.14) \qquad \sigma(x) = \theta(x) + \frac{1}{\pi} \,\mathrm{tg}\, \mu\pi \; x^{-\mu} \fint_0^{\infty} \frac{u^{\mu+1}\theta(u)}{u^2 - x^2} \, du.$$

Hence we have found an integral representation of s(x).

$$(4.15)\quad s(x) = \cos^2 \mu\pi.\psi(x) + \frac{1}{4\pi} \sin 2\mu\pi\ x^{-\mu} \fint_0^\infty \frac{t^\mu \psi(t)}{t-x}\, dt -$$

$$- \frac{1}{2\pi} \sin \mu\pi\ x^{-\mu} \int_0^\infty \frac{t^\mu \psi(t)}{t+x}\, dt + \frac{1}{2\pi} \sin 2\mu\pi\ x^{-2\mu} \fint_0^\infty \frac{t^{2\mu}\psi(t)}{t-x}\, dt +$$

$$+ \frac{1}{2\pi} \sin 2\mu\pi\ x^{-2\mu} \int_0^\infty \frac{t^{2\mu}\psi(t)}{t+x}\, dt.$$

We know that

$$F(\alpha) = \frac{2}{\pi} \int_0^\infty s(x) \sin \alpha x\, dx$$

and with the aid of this formula we find

$$(4.16)\quad \pi F(\alpha) = \sin^2 \mu\pi \int_0^\infty \psi(x) \sin \alpha x\, dx -$$

$$- \sin \mu\pi \sin \frac{\mu\pi}{2} \int_0^\infty \psi(x)\, U_{\mu+1}(2\alpha x, 0)\, dx -$$

$$- \sin \mu\pi \cos \frac{\mu\pi}{2} \int_0^\infty \psi(x)\, U_\mu(2\alpha x, 0)\, dx +$$

$$+ 2\cos \mu\pi \int_0^\infty \psi(x)\, U_{2\mu+1}(2\alpha x, 0)\, dx,$$

where U_ν again denotes Lommel's function of two variables. Depending on the form of the known function $\psi(x)$ sometimes a different representation of the answer is preferable, viz.

$$\pi F(\alpha) = 2\cos^2\mu\pi \int_0^\infty \psi(x)\sin\alpha x dx - \frac{3}{2} \sin 2\mu\pi \int_0^\infty \psi(x)\cos\alpha x dx -$$

$$- \frac{1}{\pi} \sin^2\mu\pi \sin \frac{\mu\pi}{2} \int_0^\infty \Psi(\alpha x) \frac{x^{-\mu}}{x^2+1}\, dx + \frac{1}{\pi} \sin^2\mu\pi\cos \frac{\mu\pi}{2} \int_0^\infty \Psi(\alpha x) \frac{x^{1-\mu}}{x^2+1}\, dx +$$

$$+ \frac{2}{\pi} \sin 2\mu\pi \, \cos\mu\pi \int_0^\infty \Psi(\alpha x) \, \frac{x^{-2\mu}}{x^2+1} \, dx,$$

where $\Psi(\alpha x) = \int_0^\infty e^{-\alpha x t} \, \psi(t) \, dt$ is the Laplace transform of $\psi(t)$.

Thus we have found by a fairly simple method a solution in closed form of the problem posed in section 1.

5. An example.

As an example we shall consider the potential of a point source situated in the quarter plane $x > 0$, $y > 0$. This problem is governed by the potential equation.

(5.1) $$\Delta u = -2\pi\ \delta(x-x_0)\ \delta(y-y_0), \qquad x,x_0 > 0,\ y,y_0 > 0,$$

where δ denotes the Dirac delta function.
We impose the boundary conditions

(5.2)
$$u = 0, \qquad x = 0,\ y > 0,$$
$$\frac{\partial u}{\partial y} + k\,\frac{\partial u}{\partial x} = 0, \qquad x > 0,\ y = 0,$$

where $k = \operatorname{tg} \mu\pi$, $-1/2 \le \mu \le 0$, is a known real constant.

To solve the problem we write

(5.3) $$u = -\ln|z^2-z_0^2| + \ln|z^2-\bar{z}_0^2| + u^{(1)},$$

then (5.1) and (5.2) are satisfied when

$$\Delta u^{(1)} = 0, \qquad x,y > 0,$$

$$u^{(1)} = 0, \qquad x = 0,\ y > 0,$$

and $$\frac{\partial u^{(1)}}{\partial y} + k\,\frac{\partial u^{(1)}}{\partial x} = -\psi(x), \qquad x > 0,\ y = 0,$$

(5.4) where $$\psi(x) = -i\left(\frac{1}{x-z_0} + \frac{1}{x+z_0} - \frac{1}{x-\bar{z}_0} - \frac{1}{x+\bar{z}_0}\right).$$

Again we substitute

$$u^{(1)} = \int_0^\infty \alpha^{-1}\, f(\alpha)\,\sin\alpha x\, e^{-\alpha y}\, d\alpha,$$

then Laplace's equation and the first boundary condition are satisfied, while $f(\alpha)$ can be solved from the second condition which takes the form (1.5). Using (2.6) we obtain

$$(5.5)\quad \Psi(z) = \frac{2}{\pi i}\, e^{\mu\pi i} \cos\mu\pi\, z^{-2\mu} \int_0^\infty \frac{s^{2\mu+1}}{s^2-z^2} \left(\frac{1}{s-z_0} + \frac{1}{s+z_0} - \frac{1}{s-\bar{z}_0} - \frac{1}{s+\bar{z}_0}\right) ds.$$

We evaluate this integral by integrating along the contour shown in figure 1. The contribution of the large circle tends to zero as the radius of the circle tends to infinity.

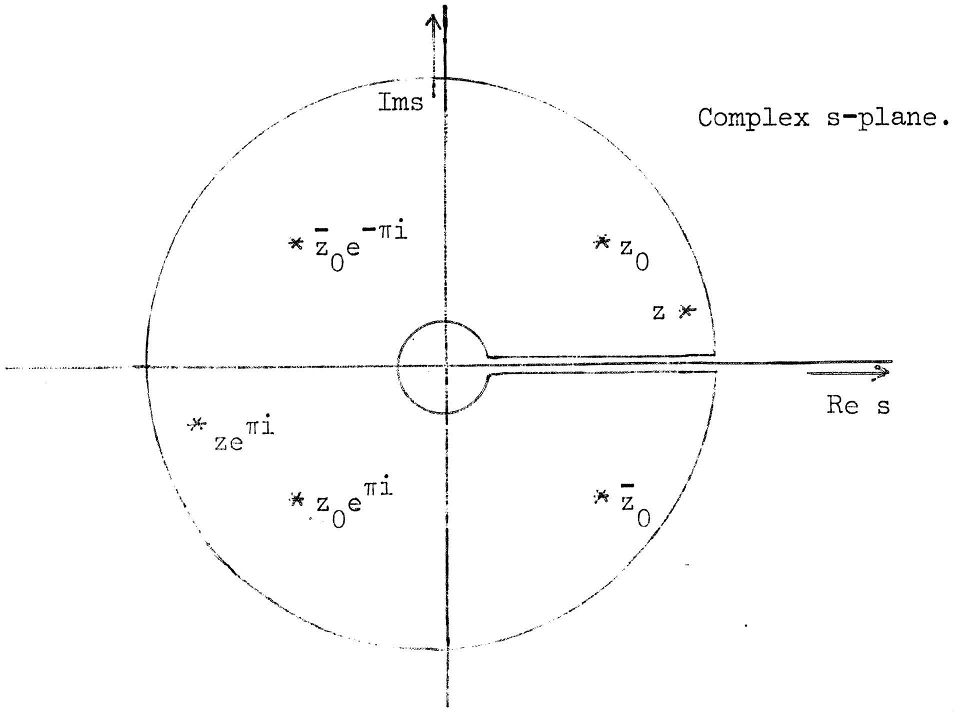

Figure 1.

As a consequence of the cut along the positive real axis in the s-plane we must distinguish between $0 < \arg z < \pi$ and $\pi < \arg z < 2\pi$.
In the first case

$$\frac{1}{2\pi}\left(1-e^{4\mu\pi i}\right) \int_0^\infty \frac{s^{2\ +1}}{s^2-z^2}\, \psi(s)ds = -4ie^{\mu\pi i}\sin\mu\pi\, z^{2\mu+1}\left(\frac{1}{z^2-z_0^2} - \frac{1}{z^2-\bar{z}_0^2}\right) + $$
$$+ 4ie^{\mu\pi i}\sin\mu\pi\, \frac{z_0^{2\mu+1}}{z^2-z_0^2} + 4ie^{-\mu\pi i}\sin\mu\pi\, \frac{\bar{z}_0^{2\mu+1}}{z^2-\bar{z}_0^2}$$

and in the second case

$$= 4ie^{-\mu\pi i}\sin\mu\pi\ z^{2\mu+1}\left(\frac{1}{z^2-z_0^2} - \frac{1}{z^2-\bar{z}_0^2}\right) + 4ie^{\mu\pi i}\sin\mu\pi\ \frac{z_0^{2\mu+1}}{z^2-z_0^2} +$$

$$+ 4ie^{-\mu\pi i}\sin\mu\pi\ \frac{\bar{z}_0^{2\mu+1}}{z^2-\bar{z}_0^2} .$$

The real axis in the z-plane is a line of discontinuity. The points $z = z_0,\ z_0e^{\pi i},\ \bar{z}_0,\ \bar{z}_0e^{-\pi i}$ however are no poles as would appear from a superficial look at the formulae.

We confine ourselves to the area $0 < \arg z < \pi$ where

$$(5.6)\quad \Psi(z) = \frac{2z}{z^2-z_0^2} - \frac{2z}{z^2-\bar{z}_0^2} - 2z_0^{2\mu+1}\frac{z^{-2\mu}}{z^2-z_0^2} - 2e^{-2\mu\pi i}\,\bar{z}_0^{2\mu+1}\frac{z^{-2\mu}}{z^2-\bar{z}_0^2}$$

and in the limiting case

$$\mathrm{Re}\{\Psi^+(x)\} = -4\cos\mu\pi\ x^{-2\mu}\ \mathrm{Re}\{e^{-\mu\pi i}\frac{z_0^{2\mu+1}}{x^2-z_0^2}\} .$$

According to (2.10)

$$(5.7)\qquad u^{(1)} = \frac{i}{2\pi}\int_0^\infty \mathrm{Re}\{\Psi^+(t)\}\ [\ln(t^2-z^2) - \ln(t^2-\bar{z}^2)]\ dt.$$

We first consider the following integral

$$I = \int_{-\infty}^{\infty} \Psi(t)\ [\ln(t^2-z^2) - \ln(t^2-\bar{z}^2)]\ dt$$

and determine it by integrating along the contour of figure 2. Each of the logarithms is associated with two branch points in the complex t-plane, respectively $t = z,\ ze^{\pi i}$ and $t = \bar{z},\ \bar{z}e^{-\pi i}$. The integral over the large half circle tends to zero as the radius of the circle tends to infinity.

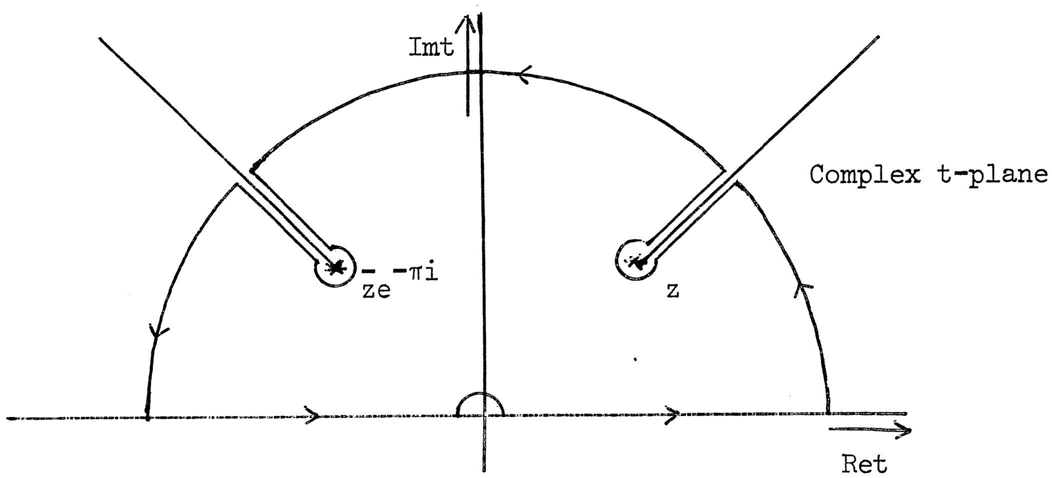

Figure 2.

Some manipulation shows that

$$I = 2\int_0^\infty \mathrm{Re}\{\Psi^+(t)\}\ [\ln(t^2-z^2) - \ln(t^2-\bar{z}^2)]\ dt,$$

i.e. the integral we need to determine (5.7) and also

$$I = \int_z^{\infty e^{i\phi}} \Psi(t)\ [\ln(t-z) - \ln e^{-2\pi i}(t-z)]\ dt -$$

$$- \int_{\bar{z}e^{-\pi i}}^{\infty e^{i(\pi-\phi)}} \Psi(t)\ [\ln(t+\bar{z}) - \ln e^{-2\pi i}(t+\bar{z})]\ dt$$

$$= 2\pi i\ \left[\int_z^{\infty e^{i\phi}} - \int_{\bar{z}e^{-\pi i}}^{\infty e^{i(\pi-\phi)}}\right]\ \Psi(t)\ dt.$$

From (5.6) we see that the integrals over the first two terms of $\Psi(t)$ are of the type

$$\left[\int_z^{\infty e^{i\phi}} - \int_{\bar{z}e^{-\pi i}}^{\infty e^{i(\pi-\phi)}}\right] \frac{tdt}{t^2-\xi^2} = -\int_{\bar{z}e^{-\pi i}}^{z} \frac{tdt}{t^2-\xi^2} = -\frac{1}{2}\ln(z^2-\xi^2) + \frac{1}{2}\ln(\bar{z}^2-\xi^2).$$

The other integrands are of the type

$$\frac{\xi^{2\mu+1}t^{-2\mu}}{t^2-\xi^2} = \frac{(\xi/t)^{2\mu}}{2\xi}\ \left[\frac{1}{(t/\xi)-1} - \frac{1}{(t/\xi)+1}\right]$$

and the best results are obtained by expanding $(\xi/t)^{2\mu}$ in terms of the corresponding denominator, e.g.

$$(\xi/t)^{2\mu} = \left[(t/\xi)-1+1\right]^{-2\mu} = \sum_{n=0}^{\infty} \binom{-2\mu}{n} \left[(t/\xi)-1\right]^{n}, \text{ if } |(t/\xi)-1| \leq 1,$$

$$= \sum_{n=0}^{\infty} \binom{-2\mu}{n} \left[(t/\xi)-1\right]^{-2\mu-n}, \text{ if } |(t/\xi)-1| \geq 1.$$

The choice of the point of observation in $x > 0$, $y > 0$ determines which expansions must be used as is shown in figure 3.

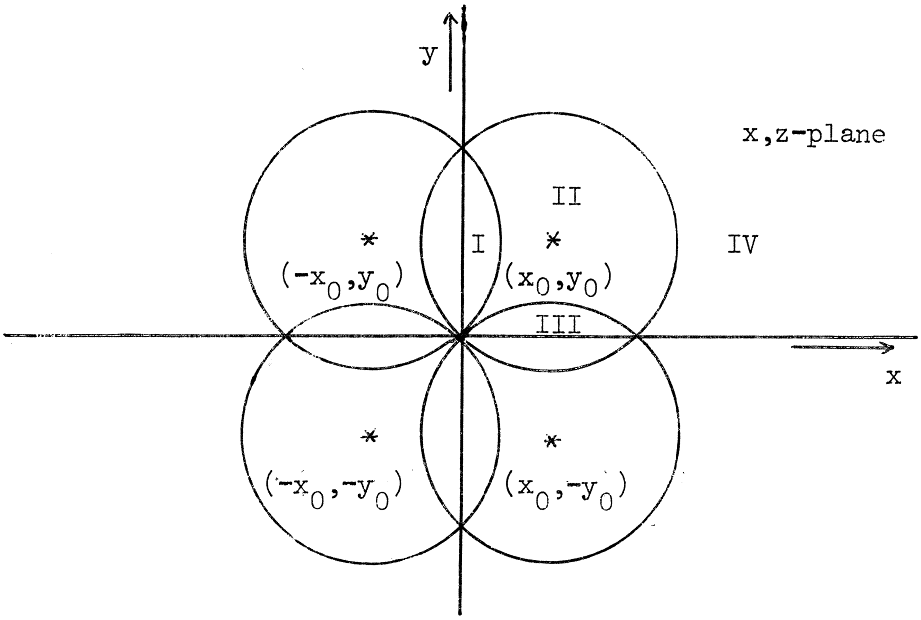

Figure 3.

If $|(t/\xi)-1| \leq 1$ or $|(t/\xi)-1| \geq 1$ along all of the line parallel to the real t-axis connecting $t = \bar{z}e^{-\pi i}$ and $t = z$ we close the contour using this line and the arc of a circle at infinity.

$$\left[\int_{z}^{\infty e^{i\phi}} - \int_{\bar{z}e^{-\pi i}}^{\infty e^{i(\pi-\phi)}}\right] \frac{(\xi/t)^{2\mu}}{t-\xi}\, dt =$$

$$= \ln(\bar{z}e^{-\pi i}-\xi)-\ln(z-\xi) + \sum_{n=1}^{\infty} \binom{-2\mu}{n} \frac{\xi^{-n}}{n} \left[(\bar{z}e^{-\pi i}-\xi)^{n}-(z-\xi)^{n}\right],$$

in the first case.

$$= - \sum_{n=0}^{\infty} \binom{-2\mu}{n} \frac{\xi^{2\mu+n}}{2\mu+n} \left[(\bar{z}e^{-\pi i}-\xi)^{-2\mu-n} - (z-\xi)^{-2\mu-n}\right],$$

in the second case.

In all other cases we close the contour using the arc of a circle at infinity combined with a curve connecting $t = \bar{z}e^{-\pi i}$ and $t = z$ in such a way that either we can use one type of expansion on all of the curve or the curve is divided in two parts at a convenient point ($t = 0$) so that on each of the parts one type of expansion holds. The resulting integrals are of the same type as above.
We finally obtain

$$(5.8)\quad u^{(1)} = \ln\left|\frac{z+z_0}{z-\bar{z}_0}\right| - \sum_{n=1}^{\infty} \binom{-2\mu}{n} \frac{1}{n} \mathrm{Re}\{(\frac{z-z_0}{z_0})^n - (-1)^n (\frac{z+\bar{z}_0}{\bar{z}_0})^n\} -$$

$$- \sum_{n=0}^{\infty} \binom{-2\mu}{n} \frac{1}{2\mu+n} \mathrm{Re}\{(-1)^n (\frac{z+z_0}{z_0})^{-2\mu-n} - e^{-2\mu\pi i} (\frac{z-\bar{z}_0}{\bar{z}_0})^{-2\mu-n}\},$$

$(x,y) \in \mathrm{I}$,

$$(5.9)\quad u^{(1)} = \ln\left|\frac{(z+z_0)z_0}{z^2-\bar{z}_0^2}\right| + \pi \operatorname{cotg} 2\mu\pi - \sum_{n=1}^{\infty} \binom{-2\mu}{n} \frac{1}{n} \mathrm{Re}\,\{(\frac{z-z_0}{z_0})^n - (-1)^n\} -$$

$$- \sum_{n=0}^{\infty} \binom{-2\mu}{n} \frac{1}{2\mu+n} \mathrm{Re}\{(-1)^n e^{-2\mu\pi i} (\frac{z+\bar{z}_0}{\bar{z}_0})^{-2\mu-n} +$$

$$+ (-1)^n (\frac{z+z_0}{z_0})^{-2\mu-n} - e^{-2\mu\pi i} (\frac{z-\bar{z}_0}{\bar{z}_0})^{-2\mu-n}\},$$

$(x,y) \in \mathrm{II}$,

$$(5.10)\quad u^{(1)} = \ln\left|\frac{(z+z_0)z_0}{z^2-\bar{z}_0^2}\right| - \mathrm{Re}\{e^{-2\mu\pi i} \ln(\frac{z-\bar{z}_0}{\bar{z}_0})\} + \pi(\operatorname{cotg} 2\mu\pi - \sin 2\mu\pi) -$$

$$- \sum_{n=1}^{\infty} \binom{-2\mu}{n} \frac{1}{n} \mathrm{Re}\{(\frac{z-z_0}{z_0})^n + e^{-2\mu\pi i} (\frac{z-\bar{z}_0}{\bar{z}_0})^n - (-1)^n(1+\cos 2\mu\pi)\} -$$

$$- \sum_{n=0}^{\infty} \binom{-2\mu}{n} \frac{1}{2\mu+n} \mathrm{Re}\{(-1)^n e^{-2\mu\pi i} (\frac{z+\bar{z}_0}{\bar{z}_0})^{-2\mu-n} +$$

$$+ (-1)^n (\frac{z+z_0}{z_0})^{-2\mu-n} - (-1)^n\},$$

$(x,y) \in \mathrm{III}$,

$$(5.11)\quad u^{(1)} = \ln\left|\frac{z^2-z_0^2}{z^2-\bar{z}_0^2}\right| - \sum_{n=0}^{\infty} \binom{-2\mu}{n} \frac{1}{2\mu+n} \mathrm{Re}\Big\{-\left(\frac{z-z_0}{z_0}\right)^{-2\mu-n} +$$

$$+ (-1)^n e^{-2\mu\pi i} \left(\frac{z+\bar{z}_0}{\bar{z}_0}\right)^{-2\mu-n} +$$

$$+ (-1)^n \left(\frac{z+z_0}{z_0}\right)^{-2\mu-n} - e^{-2\mu\pi i} \left(\frac{z-\bar{z}_0}{\bar{z}_0}\right)^{-2\mu-n}\Big\}, \qquad (x,y) \in IV.$$

It can be ascertained that the boundary condition for $x = 0$, $y > 0$ is satisfied by (5.8) and (5.11) while the condition for $x > 0$, $y = 0$ is satisfied by (5.10) and (5.11) (see figure 3).

6. Appendix.

Another way of solving this problem is by using the general method developed by LAUWERIER [3], although this implies that we cannot make an optimal use of the simple geometry of the problem. The solution obtained in this way shows obvious similarities with the solution of section 4. However it is difficult to show the equivalence of both forms. Similar difficulties were encountered by LAUWERIER [4] in the comparison of his results with those obtained by Peters.

We will just state the different formulae. With a small change in notation we write (1.3) as

$$u = \int_0^\infty g(w)\sin(x\,\mathrm{sh}\,w)\,e^{-y\,\mathrm{ch}w}\,dw$$

and
$$g(w) = \frac{2}{\pi}\int_0^\infty s(t)\sin(t\,\mathrm{sh}\,w)\,dt,$$

where $s(t)$ is given in (4.15) as

$$s(t) = \cos^2\mu\pi.\psi(t) + \frac{1}{4\pi}\sin 2\mu\pi\, t^{-\mu} \,⨍_0^\infty \frac{s^\mu\psi(s)}{s-t}\,ds -$$

$$-\frac{1}{2\pi}\sin\mu\pi\, t^{-\mu}\int_0^\infty \frac{s^\mu\psi(s)}{s+t}\,ds + \frac{1}{2\pi}\sin 2\mu\pi\, t^{-2\mu}\,⨍_0^\infty \frac{s^{2\mu}\psi(s)}{s-t}\,ds +$$

$$+\frac{1}{2\pi}\sin 2\mu\pi\, t^{-2\mu}\int_0^\infty \frac{s^{2\mu}\psi(s)}{s+t}\,ds.$$

With Lauwerier's method we get

$$g(w) = \frac{1}{\pi}\cos\mu\pi\left[\frac{\mathrm{ch}w}{\mathrm{ch}(w+\mu\pi i)} + \frac{\mathrm{ch}w}{\mathrm{ch}(w-\mu\pi i)}\right]\int_0^\infty \psi(t)\sin(t\mathrm{sh}w)\,dt +$$

$$+\frac{1}{\pi^2}\mathrm{ch}w\, K(w)\,⨍_{-\infty}^\infty \frac{\bar\psi(t)}{K(t)}\left[\frac{\mathrm{ch}t}{\mathrm{ch}(t+\mu\pi i)} - \frac{\mathrm{ch}t}{\mathrm{ch}(t-\mu\pi i)}\right]\frac{dt}{\mathrm{sh}t-\mathrm{sh}w},$$

where
$$\bar\psi(t) = \int_0^\infty \psi(s)\sin(s\,\mathrm{sh}t)\,ds,$$

$$K(w) = \frac{e\left[w+\pi i/2,(\mu-1/2)\pi\right]}{e\left[-w+\pi i/2,-(\mu-1/2)\pi\right]}$$

and

$$e(w,\gamma) = \exp\left[\frac{1}{2}\int_{-\infty}^{\infty}\frac{1-\cos tw}{t}\,\frac{\mathrm{sh}\gamma t}{\mathrm{sh}\pi t\ \mathrm{sh}\frac{\pi t}{2}}\,dt\right].$$

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